N-ification of forces: a holographic perspective on D-brane model building

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ABSTRACT: We study geometric aspects of extensions of the supersymmetric standard model that exhibit a periodic duality cascade. Via the holographic correspondence, the growth of the gauge group rank towards the UV is interpreted as a gradual decompactification transition. We show that this class of models typically develop a duality wall in the UV, and present an efficient method for estimating the hierarchy between the on-set of the cascade and the formation of the wall. As an illustrative example, we study the model introduced by Cascales, Saad and Uranga in [14], which has an known geometric realization in terms of D-branes on an SPP/$\mathbb{Z}_3$ singularity.

Contents

1 Introduction

Supersymmetric Grand Unification still stands out as the most compelling paradigm for beyond the Standard Model physics — it is elegant, explains the value of $\sin\theta_W$, and can be naturally embedded in string theory [1].[1] Several developments within string theory, however, have motivated the exploration of an array of alternative possibilities. In particular, the discoveries of D-branes [3], warped flux compactifications, and the string landscape [4] have led to the recognition that supersymmetric GUTs form only a very small subset among many conceivable string realizations of the Standard Model.

A common feature of all string (motivated) phenomenological scenarios is that 4-d effective field theory breaks down at some high energy scale, above which extra dimensional or string physics takes over. Traditionally, the decompactification scale is taken to be close to the GUT scale, but in many popular scenarios, it is assumed to be much lower. From the

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[1]See [2] for a review of the current status.

4-d vantage point, the decompactification transition manifests itself via a sudden escalation in the number of field theoretic degrees of freedom. Any well thought out phenomenological scenario based on extra dimensions must therefore supply a cogent description of the dynamics among all these extra dimensional modes, and of their interactions with Standard Model particles. A convincing way of specifying these dynamical rules is by finding a natural realization of the given extra dimensional scenario within string theory.

A popular class of low scale scenarios postulates that the Standard Model degrees of freedom reside inside a 5-d warped geometry, either localized on branes or more smoothly distributed along the extra dimension [5]. An attractive aspect of this type of models is that the warped extra dimensional physics can be viewed through the prism of the AdS/CFT correspondence [6–8], and reformulated in pure 4-d language in terms of the dual strongly coupled large N gauge theory. The AdS/CFT dictionary comes with precise rules, which, if followed correctly, supply a consistent string embedding of the given warped scenario.

Via the holographic perspective, the decompactification transition is recast as a sudden expansion of the total effective rank N (or number of 'colors') of the 4-d gauge theory. This precipitous increase in the number of colors may arise via symmetry restoration or deconfinement of a large N gauge group. Models with large warped extra dimensions are extreme representatives of a continuous family of beyond-the-Standard-Model physics scenarios in which the total rank increases with energy scale.

A natural way for the number of colors to run with scale is via a duality cascade [9, 10]. In the context of renormalization group flows, a change in effective rank arises when an asymptotically free gauge theory flows to strong coupling in the IR, where its low energy physics allows a weakly coupled description in terms of a dual theory with fewer colors. A duality cascade is the general phenomenon whereby a quiver gauge theory undergoes a sequence of Seiberg dualities. Cascading RG flows are of particular interest because they occur for many field theories with known supergravity duals [9, 11, 12]. The classic example is the Klebanov-Strassler (KS) gauge theory, the holographic dual to the fluxed deformed conifold [9].

The existence of a holographic dual is directly linked to the realization of the gauge theory via open strings attached to a stack of D-branes. Starting with just a few D-branes in the IR, near the bottom of the warped throat, the cascade will drive the effective number of branes to increase towards the UV, until it is large enough to be captured by a dual geometry.

As noted by several authors [13–15], the existence of duality cascades suggests an interesting new avenue for model building. In the IR we observe the Standard Model, which has a finite rank gauge group. In the UV, on the other hand, string theory motivates the possibility that physics becomes higher dimensional and may admit, over some energy range, a dual description in terms of a large N 4-d gauge theory. It is then natural to consider the general class of scenarios in which the Standard Model in the IR and the large N theory in the UV are smoothly connected. We will refer to this class of new physics models as 'N-ification scenarios'.

N-ification: decompactification via a duality cascade. As indicated in figure 1, N-ification scenarios in a sense interpolate between Grand Unified models, that assume a relatively minimal field content below the GUT scale, and large/warped extra dimensional scenarios [5, 16], that postulate the appearance of many new degrees of freedom at a relatively low scale. In N-ification models, the the decompactification process starts at a low scale, but takes place gradually.

There are many possible ways in which the MSSM may connect to a cascading gauge theory in the UV. Starting in the IR, the first step upwards in the cascade introduces extra gauge and matter degrees of freedom. These extra particles are invisible below some scale Λ_c, at which they become confined or acquire a mass via symmetry breaking. For any realistic scenario, the experimental limits on compositeness or existence of exotic matter dictate a lower bound for the critical scale Λ_c in the multi-TeV range.

Given the first step upwards in the cascade, and assuming that no other new gauge groups appear at some higher scale, the UV physics in principle follows from integrating the RG flow upwards. Whenever one of the gauge groups reaches strong coupling, one can use Seiberg duality to switch to a weakly coupled electric formulation. Without extra guidance from string theory or some other UV-motivated principle, however, the bottom-up perspective typically leads to irregular duality sequences, that involve rapidly growing ranks and numbers of generations. A more attractive class of models are the periodic scenarios, in which the structure of the quiver diagram and the number of generations remain constant throughout the cascade, and only the ranks of the gauge groups change. Besides being more appealing as field theories, such regularly cascading theories are also more likely to allow a realization as the world-volume theory of D-branes at some suitable geometric singularity.

In the KS gauge theory, the duality sequence unfolds evenly and the number of colors grows logarithmically with RG scale. In general, however, and in particular for all known cascade extensions on the MSSM, the duality steps tend to accumulate in shorter and shorter intervals, and the gauge group ranks grow at an increasingly rapid pace, as the systems flows towards the UV. Eventually the theory gets trapped inside a strong coupling regime and hits a so-called duality wall, located at some finite UV scale [11, 13, 15, 17, 18]. At the wall, the rank and 't Hooft coupling of the gauge theory diverge and the gauge theory description breaks down. Cascading RG flows, even without duality walls, are inherently strongly coupled. As a result, their field theory analysis is subtle and one should be cautious about its interpretation. The most rigorous support for the field theory conclusions comes from supergravity duals (for those theories that have them). This is precisely the case for the model we consider in this paper. As we will show, a duality wall arises from the logarithmic running of the total gauge coupling (which is computed from the dilaton profile in supergravity) by dimensional transmutation. In a complete framework, the physics near the duality wall is assumed to be regulated via the decompactification transition from the 4-d gauge theory to a full-fledged higher dimensional string compactification.

Igniting the cascade: deconstructed decompactification. It can be shown under quite general assumptions that the embedding of the MSSM inside a cascading gauge
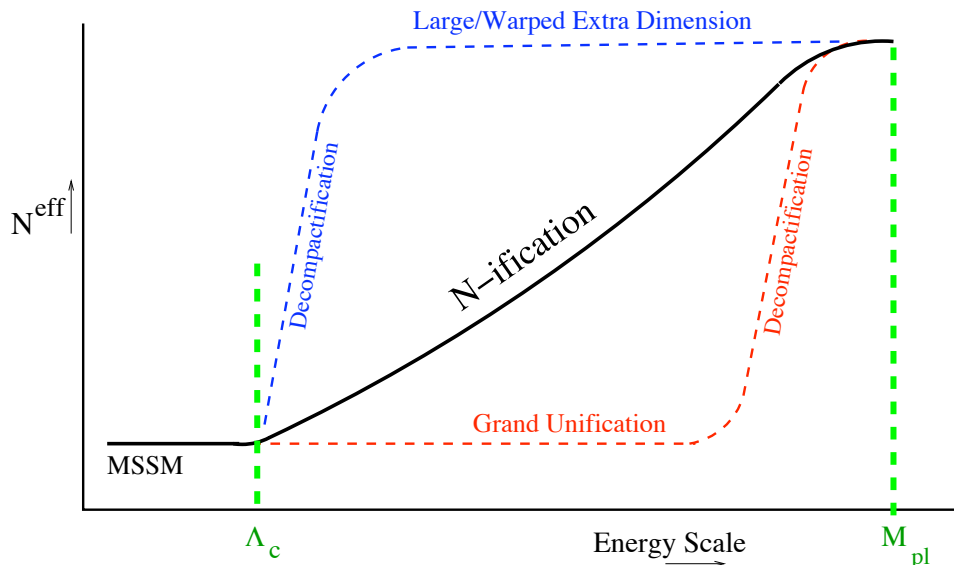
Figure 1. The number of degrees of freedom as a function of energy scale for three scenarios. In Grand Unified models, the MSSM fields (possibly with some messengers) capture all visible matter up to the GUT scale. In low scale extra dimension scenarios, the degrees of freedom proliferate at a decompactification scale $\Lambda_c \ll M_{\mathrm{GUT}}$. In N-ification, extra degrees of freedom appear gradually via a duality cascade, that starts at Λ_c and accelerates towards the UV.

theory requires the introduction of one or more extra gauge groups. The decoupling transition, at the scale Λ_c, of this extra sector from the MSSM can take place via three basic mechanisms [15]:

Confinement. The extra gauge group factor G has fewer flavors than colors, and becomes strongly coupled at some energy scale Λ_c. Matter particles charged under G confine and pair up into mesons, which manifest themselves as MSSM matter in the IR.

Higgsing. Bifundamental matter connecting two quiver nodes develops an expectation value at some scale Λ_c, breaking a product gauge group to its diagonal. The unbroken gauge symmetry is part of the MSSM-gauge group.

Decoupling. Bifundamental matter connecting two quiver nodes has a mass equal to Λ_c. Below the scale Λ_c, the extra gauge sector is completely decoupled from the MSSM.

A general N-ification scenario may involve a combination of these three mechanism. Moreover, extra matter could in principle continue to show up at higher energy scales beyond Λ_c. To keep things minimal, however, we will imagine that, once the RG cascade has been initiated, no other new gauge groups will appear. The UV physics then in principle follows from integrating the RG flow upwards, and by using the rules of Seiberg duality whenever one of the gauge group factors reaches strong coupling.

We would like to think of the appearance of the extra matter at Λ_c as are the first indication of extra dimensional physics. From this perspective, an attractive realization of a gauge theory one step up in the duality cascade is shown in figure 2. It depicts a minimal
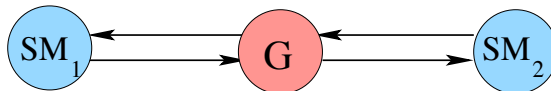
Figure 2. The quiver diagram above the last step of the duality cascade. The gauge group G has $N_f \leq N_c$ and confines at a scale Λ_c. The associated mesons acquire a vev that breaks the product of two Standard Model gauge groups to the diagonal.

realization of the MSSM extended by a deconstructed small extra dimension [19]. The extra gauge group G is connected via vector pairs of bifundamental matter to two copies of the Standard Model. We assume that the number of bifundamentals is less than the rank of G, so that the gauge group G has $N_f \leq N_c$ and confines at some scale Λ_c. The extra charged matter pairs up to form mesons. The strong gauge dynamics generates an ADS superpotential, or a quantum modified moduli space in the case that $N_f = N_c$. As a result, some mesons get non-zero vacuum expectation values, that, in a proper set-up, break the product of two Standard Model gauge groups to the diagonal. Most of the mesons acquire a mass of order Λ_c and decouple; some may remain light and produce part the MSSM spectrum.

This particular realization of the first cascade step thus involves a combination of the confinement and Higgsing scenario. It reveals the basic characteristics of a decompactification transition of a small extra dimension [19]: above the symmetry breaking mass scale set by the meson vevs, the theory exhibits a KK doubling of the MSSM degrees of freedom. Below Λ_c only the constant modes survives, leaving only the MSSM in the infrared.

Deconstruction produces only a rudimentary version of extra dimensional physics. In particular, it does not include 5-d gravity. However, once the deconstructed dimension has been created, the N-ification scenario assumes that the rest of the extra dimensional dynamics emerges via the RG cascade and the holographic correspondence. Above the deconfinement transition of G, the new bifundamental matter accelerates the RG running of the two MSSM factors. Integrating upwards, the MSSM gauge groups hit strong coupling at an intermediate scale, considerably above Λ_c but much below M_{Pl}. This is where the next Seiberg duality takes place. This process continues until at some high energy scale all gauge groups have large rank and large 't Hooft coupling, and the holographic AdS/CFT correspondence becomes applicable. The transition at Λ_c can thus be thought of as the first stage of a gradual decompactification process.

The energy scale above which the closed string description becomes accurate depends on detailed parameters, such as the rank of the gauge group G at the bottom of the cascade. A priori, we may choose G to have a large rank. This gives rise to a low scale extra dimensional scenario. In the following, however, we will mostly consider the case in which G has relatively small rank, and the large N regime appears farther towards the UV — with the wall ideally close to or somewhat below the Planck scale.

Hierarchy of scales.   Experience shows that cascading UV extensions of the MSSM typically display an accelerated sequence of cascade steps that eventually hit a duality wall at some high scale. We imagine that upon embedding inside a complete string compactification, the duality wall is regulated via the transition towards the full 10-dimensional theory. Each specific N-ification scenario thus comes with a rather precise prediction for the hierarchy between the low scale Λ_c, where the new physics first shows up, and the high scale where the theory fully decompactifies. This hierarchy depends on various discrete parameters, such as the number of families and the rank of the extra gauge group G.

Computing the hierarchy in general would seem involved or even impossibly hard, since the cascading quiver theory typically contains one or more strongly coupled gauge sectors, so that anomalous dimensions are not small. As we will show in the following, however, one can gain some partial control over the calculation by considering the total beta function, that describes the running of sum of the (inverse gauge couplings)^2

$$x_{\rm tot} = \sum_i x_i \,, \qquad\qquad x_i = \frac{8\pi^2}{g_i^2} \,. \qquad\qquad (1.1)$$

This total gauge coupling is relatively insensitive to the duality transitions, and behaves smoothly until it hits a strong coupling singularity $x_{\rm tot} = 0$ at the location of the duality wall. We will find that, once the theory reaches a regime where the supergravity dual description becomes applicable, the scale dependence of $x_{\rm tot}$ becomes identified with the profile of the dilaton along the holographic radial direction. By matching this dilaton profile with the behavior of $x_{\rm tot}$ in the gauge theory regime, one can make a reasonably accurate estimate of the hierarchy of scales between the scale Λ_c and the scale where the duality wall sets in.

Figure 3 gives a schematic depiction of a typical profile of $x_{\rm tot}$. Below the scale Λ_c, it follows the total coupling of the MSSM. Just above Λ_c, the gauge theory cascade sets in. Initially, in regime I, the ranks of the gauge groups are not yet large enough to admit an controlled string dual description. At the intermediate range II, all the gauge group ranks are large enough so that the dual supergravity description becomes valid. Finally, near regime III, the duality wall appears and the transition to the full 10-dimensional string theory takes place.

An explicit example.   The N-ification program can in principle be formulated and developed from a purely field theoretic perspective. Indeed, one may view it as an approach to string phenomenology that, rather than looking for ways to derive particle physics from string theory, looks to develop scenarios for which the UV string theory can be derived from the field theory. To find geometrically attractive realizations, however, it is natural to look for guidance from string theory. So we adopt the hypothesis that the Standard Model, and its UV extension in the form of a cascading gauge theory, both arise as the low energy effective field theory of open strings attached to a configuration of D-branes, placed within some suitable geometry.

As in [9], we are interested in finding periodic examples, for which the quiver diagram returns to itself after a few Seiberg duality steps, but with a new assignment of gauge
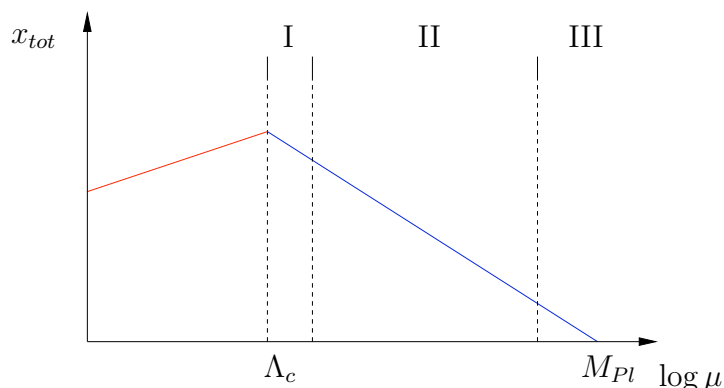
Figure 3. The typical behavior of the total inverse gauge coupling x_{tot} as a function of scale. The three regimes indicated are: (I) the onset of the RG cascade in the IR, (II) the large N regime in which the dual supergravity description is valid, (III) the duality wall where the transition to the full 10-d string theory takes place.

groups ranks. For such periodic cascades, the 4-d quiver gauge theory reaches a smooth large N limit, for which it may be easier to identify a dual string description.

An interesting cascading extension of an MSSM-like theory was introduced and studied in [14]. The quiver theory has the structure shown in figure 2, where upon evolving the cascade towards the UV, the two light-blue nodes turn into large N versions of the Standard Model gauge group $U(N + 3) \times U(N + 2) \times U(N + 1)$. The model was shown to admit an explicit string theory realization in terms of D3 and D7-branes, placed on a $\mathbb{Z}_3$ orbifold of a particular Calabi-Yau singularity, known as the cone over the suspended pinch point. The theory has the hierarchical structure shown in figure 3. In the large N region II, it has a known supergravity dual that develops a duality wall near the Planck scale.[2] In the infrared, it reduces to a semi-realistic field theory, that shares most essential features (gauge group, chiral matter content, and couplings) of the MSSM. We will center our presentation on this particular example, as it provides a useful and attractive illustration of our general ideas.[3] We would like to reiterate, however, that our general perspective can also be applied to pure field theoretic models, that may reproduce low energy phenomenology in greater detail.

Outline of the paper. This paper is organized as follows. In sections 3 to 4, we review the particular embedding of the MSSM quiver theory inside of duality cascade introduced in [14]. Our exposition is somewhat schematic, since most technical aspects of the model are carefully worked out and explained in the original paper [14]. We will attempt to motivate

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[2]Near the IR end of region II, energies are comparable to Λ_c. Because of this, the effect of the complex deformation dual to the strong dynamics of G is important. Despite being conceptually equivalent to KS, finding an explicit gravity dual for this region (based on the deformed cone) is formidable task due to the small isometry group. In section 6, we present the gravity dual based on the singular cone, which is valid at energies well above Λ_c. This solution is analogous to the Klebanov-Tseytlin one for the conifold [21].

[3]Our choice of flavor D7-branes differs slightly from the one in [14], but their analysis applies with trivial modifications.

the construction and highlight various physical aspects. The quiver theory associated with an SPP singularity is introduced in section 2, as an example of a deconstructed small extra dimension. The MSSM-like quiver theory an its D-brane realization is summarized in section 3; some more detailed discussions are referred to appendix A. Sections 4 and 5 study the RG cascade. We discuss the calculation of the running of gauge couplings in strongly coupled cascading theories and present the computation of the hierarchy of scales between the onset of the cascade and the location of the duality wall. We will find that the UV scale is roughly 12 orders of magnitude above Λ_c.

In section 6 we explain some general properties of supergravity duals of D-brane gauge theories with flavor D7-branes. In particular, we explain how the presence of the D7-branes naturally leads to an accelerated RG cascade, with a non-trivial dilaton profile and a duality wall. We compute the total beta-function, that governs the radial dependence of the dilaton, and show that it is proportional to the number of flavor branes. The precise relation naturally follows from the fact that the dilaton is paired with the axion, which in turn is sourced by the D7-branes. We show that the supergravity result agrees with the gauge theory calculation.

2 A deconstructed small extra dimension

In this section, we present the geometric realization of a three node quiver, that takes the form of a deconstructed extra dimension of the type shown in figure 2. As a warm-up, we first consider the toy example in which the Standard Model node is replaced by a simple $U(N)$ gauge factor. The set up then involves N D-branes placed at a simple Calabi-Yau singularity known as the suspended pinch point (SPP) singularity.

2.1 D-branes at the suspended pinch point

When placed in the proximity of a Calabi-Yau singularity, D-branes typically rearrange themselves into so-called fractional branes, D-brane bound states wrapped around small compact cycles supported within the singularity. The gauge theory on a collection of such branes takes the form of a quiver gauge theory, where each stack of N fractional branes represents a separate $U(N)$ quiver node. The open strings at the intersections between the branes give rise to massless bifundamental matter, represented by the oriented lines that connect the quiver nodes.

The suspended pinch point (SPP) singularity may be obtained via a partial resolution of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity [20]. It is described by the following equation in $\mathbb{C}^4$

$$cd = a^2 b, \qquad (a, b, c, d) \in \mathbb{C}^4. \qquad (2.1)$$

Geometrically, the singularity looks like a complex cone over a 4-d base manifold Σ_4. For our present purpose, it is sufficient to know that the singularity supports three types of fractional branes, and thus gives rise to a quiver theory with three nodes. The quiver for a specific collection of fractional branes on the SPP singularity is shown in figure 4. The two light-blue nodes indicate $U(N)$ gauge factors. The superpotential of this SPP quiver
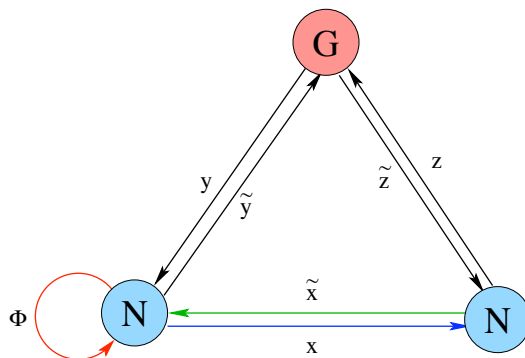
Figure 4. The three node quiver of N D3-branes and M fractional branes on an SPP singularity. The extra node with gauge group $G = \mathrm{U}(N+M)$ with $M > N$ confines in the IR.

gauge theory takes the form

$$W = \mathrm{Tr}\big(\Phi(X\tilde{X} - \tilde{Y}Y) - Z\tilde{Z}\,\tilde{X}X + \tilde{Z}ZY\tilde{Y}\big) \tag{2.2}$$

The two nodes without adjoints are fractional branes that wrap rigid cycles; the node with the adjoint corresponds to a brane that is free to move in one direction. The rank of a quiver node can be adjusted by changing the multiplicity of the corresponding fractional brane.

To realize the deconstructed extra dimension of the type shown in figure 4, we choose the multiplicity M of the third fractional brane such that the third node G has rank $N + M > 2N$. In this case, it will flow to strong coupling towards the IR and confine at some scale Λ_c. The chiral matter fields charged under G combine into mesons $M_{\mathrm{yy}} = \tilde{Y}Y$, $M_{\mathrm{yz}} = \tilde{Y}\tilde{Z}$, etc. The strong coupling dynamics gives rise to an ADS contribution to the superpotential. The superpotential then takes the form

$$W = \Phi(X\tilde{X} - M_{\mathrm{yy}}) - M_{\mathrm{zz}}\tilde{X}X + M_{\mathrm{yz}}M_{\mathrm{zy}} \; + \; (M - N)\left(\frac{\Lambda^{3M+N}}{\det \mathcal{M}}\right)^{\frac{1}{M-N}} \tag{2.3}$$

$$\mathcal{M} = \begin{bmatrix} M_{\mathrm{yz}} & M_{\mathrm{yy}} \\ M_{\mathrm{zz}} & M_{\mathrm{zy}} \end{bmatrix} = \begin{bmatrix} \tilde{Y}\tilde{Z} & \tilde{Y}Y \\ Z\tilde{Z} & ZY \end{bmatrix} . \tag{2.4}$$

The analysis simplifies slightly for the case $N = 1$. The equation of motion of M_{zy} sets $\det \mathcal{M} = \Lambda^{\frac{3M+1}{M}}$. The mesons M_{yy}, M_{yz}, M_{zy}, and the linear combination $\Phi - M_{\mathrm{zz}}$ acquire a mass. On the branch where only M_{yz} and M_{zy} have non-zero vevs, the two light-blue nodes collapse to the diagonal $\mathrm{U}(N)$ gauge theory, with three remaining light adjoint matter fields X, $\tilde{X}$ and $\Phi + M_{\mathrm{zz}}$.[4] By interpreting the two $\mathrm{U}(N)$-nodes as two points inside a small deconstructed extra dimension, the confinement transition may be viewed as the KK-reduction to the 4-d theory. The quiver with $G = \mathrm{U}(N + M)$ describes the world-volume theory of N D3 branes and M fractional branes, given by D5-branes that wrap a compact 2-cycle

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[4]We are being a bit schematic in order to simplify the discussion. Φ and M_{zz} have mass dimensions 1 and 2, respectively. Whenever we consider linear combinations of them, M_{zz} is multiplied by an implicit Λ^{-1} factor.

within the base of the SPP singularity. This set-up is analogous to the Klebanov-Strassler system, and similarly leads to a cascading gauge theory with holographic dual given by a deformation of the orginal SPP singularity supported by fluxes [22]. The passage from the D-brane theory to the smooth supergravity dual is a brane/flux geometric transition, that replaces the M fractional branes by M units of RR 3-form flux over finite size 3-cycles. The deformed geometry is described by the equation

$$cd - a^2 b = \epsilon a. \tag{2.5}$$

where ϵ is the deformation parameter. This geometry can be recovered by studying the mesonic moduli space after the confinement transition at Λ_c. The mesons represent the location of the remaining N D3-branes, moving inside the deformed SPP. The supergravity description of the deformed geometry, however, becomes accurate only if the rank of the confining node G is sufficiently large. The correspondence between gauge theory dynamics and geometry was investigated in detail in [22] and, in a context closer to this paper, in [14]. Many examples, including our main case of study, were presented in these references and we refer the reader to them for details.

In the above toy example, the deformed geometry is smooth. If the geometry after the deformation remains singular, a chiral gauge theory such as the MSSM can arise on the worldvolume of the D3-branes. In the next two sections, following [14], we will review how this toy example can be generalized into a semi-realistic UV extension of the MSSM. In this generalization, the two light-blue nodes each host one copy of the MSSM gauge group. Moreover, we will see that the three remaining light fields X, $\tilde{X}$ and $\Phi + M_{\mathrm{zz}}$ will carry the correct quantum numbers to be identified with three MSSM matter generations in the IR.

3 An MSSM-like quiver

Because the MSSM takes active part in the duality cascade, we need to embed it inside a quiver theory with somewhat special characteristics that allow the gauge group ranks to increase with every cascade step, without changing the quiver data. These special properties are naturally implied from the geometric rules of D-brane engineering.

The specific MSSM-like quiver theory that we will use in our main example is depicted in figure 5. The three light-blue colored nodes represent the three Standard Model gauge groups, extended to

$$\mathrm{U}(3) \times \mathrm{U}(2) \times \mathrm{U}(1). \tag{3.1}$$

The model has the same chiral matter content, indicated by the oriented lines, as the MSSM, three generations of quarks and leptons (each generation has its own color), except that it has a non-minimal Higgs sector, one pair of Higgs doublets per generation. The right-handed neutrinos are also missing. The theory is free of non-abelian anomalies but has mixed U(1)/non-abelian anomalies. Only the hypercharge combination

$$Y = -Q_1 - \frac{Q_2}{2} - \frac{Q_3}{3} \tag{3.2}$$
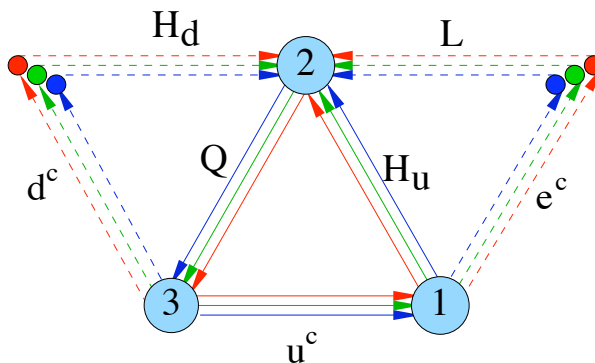
Figure 5. The MSSM quiver theory obtained from $\mathbb{C}^3/\mathbb{Z}_3$ with six fractional D-branes (light-blue nodes) and six flavor D7-branes. It describes an anomaly free gauge theory, with the matter content of the MSSM plus two extra pairs of Higgs doublets.

of abelian gauge symmetries is anomaly free and remains light. The remaining two anomalous U(1) gauge bosons are assumed to acquire a large mass, of order the string scale, via the familiar Green-Schwarz anomaly cancellation mechanism [23].

The small colored vertices on the upper left and right corner represent flavor nodes. They do not support a local gauge symmetry group. We can distinguish two different types. The three nodes on the left support chiral matter charged under U(3) × U(2); the corresponding lines are oriented such that the superpotential contains cubic Yukawa couplings $\mathrm{Tr}(\tilde{d}\, H_d\, Q)$ of the down-type quarks. The lines connected to the three flavor nodes on the right, on other other hand, are oriented in the opposite direction, forbidding the presence of cubic terms in the superpotential. Rather than with cubic Yukawa couplings, the superpotential starts out with quartic terms of the form $\mathrm{Tr}(\tilde{e}\, L\, Q\, \tilde{u})$.

The quiver theory of figure 5 can be engineered by placing D-branes on a $\mathbb{C}^3/\mathbb{Z}_3$ singularity [25]. The brane configuration involves 6 fractional branes, that wrap small compact cycles within the orbifold geometry, and give rise to the local gauge symmetry (3.1). The flavor nodes arise from D7-branes that wrap non-compact cycles. The main elements of the construction are summarized in appendix A; a more detailed exposition is found in the original literature [25].[5]

The low energy phenomenology of the MSMM-quiver model depends on the precise values of the various couplings, on the mediation mechanism for supersymmetry breaking, etc. We will not concern ourselves with these questions here, except to note that, as a world-volume theory of a local brane construction, all gauge invariant couplings (gauge and Yukawa couplings, soft parameters, etc) can in principle be freely adjusted by tuning local geometric data. Variations of the superpotential correspond to complex structure deformations of the $\mathbb{C}^3/\mathbb{Z}_3$ geometry, gauge couplings can be modified by turning on 2-form fields, and soft parameters arise by turning on F-term components of closed string fields.

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[5]See also [26–28] for other realizations of MSSM-like theories with D-branes on singularities.

4 The duality cascade

Still following [14], we now put the various ingredients together to obtain a geometric realization of a cascading extension of the MSSM-like gauge theory.

At the field theory level, the plan is to construct a gauge theory with the schematic quiver structure depicted below in figure 6. It shows the SPP quiver gauge theory, with the extra flavor nodes added 'by hand'. The two light-blue nodes represent two gauge groups that, at the bottom of the cascade, reduce to two copies of the Standard Model gauge group. The small colored nodes are the flavor nodes and the colored lines are chiral fields with the quantum numbers of the 3 generations of quarks and leptons relative to the left SM node, except that the two generations of (up)quarks are in bifundamental of the left and right SM node. In approaching the IR, the node G has $N_f < N_c$ and confines at the scale Λ_c. The corresponding mesons acquire vevs, similar as discussed for the toy model in section 2. The challenge is to find the correct structure for the gauge group G and bifundamental matter such that below the confinement transition, the effective field theory that remains in the infrared has the gauge symmetry and field content of the MSSM.

This field theoretic puzzle has a beautiful geometric solution, obtained by combining the lessons of the two previous sections [14]. The deformed SPP geometry, as specified by eq. (2.5), is invariant under the $\mathbb{Z}_3$ action

$$a \to \omega^2 a, \ b \to \omega b, \ c \to \omega c, \ d \to \omega d \tag{4.1}$$

with $\omega^3 = 1$. It is therefore possible to define the $\mathbb{Z}_3$-orbifold of the deformed SPP singularity. Thanks to the deformation, the local region near the fixed point locus of the $\mathbb{Z}_3$ action looks like the local region near the fixed point of the flat orbifold $\mathbb{C}^3/\mathbb{Z}_3$. We can thus place the same combination of fractional branes and D7-flavor branes on the deformed SPP orbifold, that produces the MSSM quiver of figure 5 when placed on the flat orbifold. The holographic correspondence then virtually guarantees that the resulting gauge theory will have the quiver structure as indicated in figure 6, and will exhibit an RG cascade that flows to the MSSM-quiver theory in the infrared.

4.1 Bottom of the cascade

The SPP/$\mathbb{Z}_3$ geometry is attractive, because the corresponding gauge theory can be easily derived by orbifolding that for the SPP. By now it is known how to derive the field theory on D-branes on arbitrary toric singularities using dimer model techniques [29]–[34]. Schematically, we know that the orbifold projection will split each of the three nodes of the SPP gauge theory into a product of three gauge groups, corresponding to the three irreducible representations of $\mathbb{Z}_3$. In [14], the Chan-Paton factors are chosen as follows

$$\begin{aligned}
\gamma_{\mathrm{SM1}} &= \mathrm{diag}\,(\,\mathbb{1}_1\,,\ \omega\mathbb{1}_2\,,\ \omega^2\mathbb{1}_3\,) \\
\gamma_{\mathrm{SM2}} &= \mathrm{diag}\,(\,\mathbb{1}_2\,,\ \omega\mathbb{1}_3\,,\ \omega^2\mathbb{1}_1\,) \\
\gamma_G &= \mathrm{diag}(\mathbb{1}_{M+3},\ \omega\mathbb{1}_{M+1},\ \omega^2\mathbb{1}_{M+2}),
\end{aligned} \tag{4.2}$$
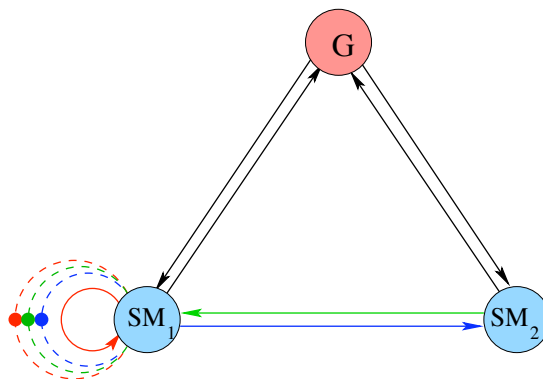
Figure 6. Schematic quiver of the cascading field theory. The light-blue nodes represent two copies of the Standard Model gauge group. The third node G confines in the IR, leading to vevs for the mesons that break the gauge symmetry to that of the Standard Model. The colored lines are chiral fields that in the IR reduce to 3 generations of quarks and leptons.

which, as promised, breaks the gauge symmetry to two copies of the Standard Model gauge group, times the extra gauge factor

$$G = \mathrm{U}(M + 1) \times \mathrm{U}(M + 2) \times \mathrm{U}(M + 3). \tag{4.3}$$

The $\mathbb{Z}_3$ transformation rule of the chiral matter fields follows from (4.1) and the identifications given in the second footnote in section 2. Using the notation as in figure 4

$$X \to \omega^2 X \quad ; \quad \tilde{X} \to \tilde{X} \quad ; \ Y \to \omega^2 Y \quad ; \quad \tilde{Y} \to \tilde{Y} \quad ;$$
$$Z \to Z \quad ; \quad \tilde{Z} \to \omega \tilde{Z}; \quad \Phi \to \omega \Phi$$

Together with the chosen assignment of Chan-Paton factors (4.2), these transformation rules dictate how the chiral matter lines connect between the quiver nodes: a field connects only those pairs of nodes for which the relative Chan-Paton phase cancels its $\mathbb{Z}_3$ charge. In a similar way, one may assign the proper Chan-Paton phases to the D7-branes, so that the corresponding flavored chiral matter fields connect to the Standard Model nodes in accordance with the MSSM-quiver of figure 5.

The resulting quiver diagram is shown in figure 7 (periodic identification of the horizontal and vertical directions is understood). Note that all nodes have an equal number of incoming and outgoing lines, ensuring that all non-abelian anomalies cancel. We have also indicated, with light dashed magenta lines, the mesons that will acquire an expectation value after confinement of the three G nodes in the middle column.

The spectrum of light particles and the symmetry breaking pattern after the confinement transition is found by following the same steps as outlined for the toy model in section 2. The superpotential of the orbifold gauge theory is inherited from the cover theory, and thus looks identical to eq. (2.2), except that all chiral fields must be generalized to carry two indices that label the two nodes that they connect to. The analysis of the orbifold spectrum is described in detail in [14], and we will not repeat it here.
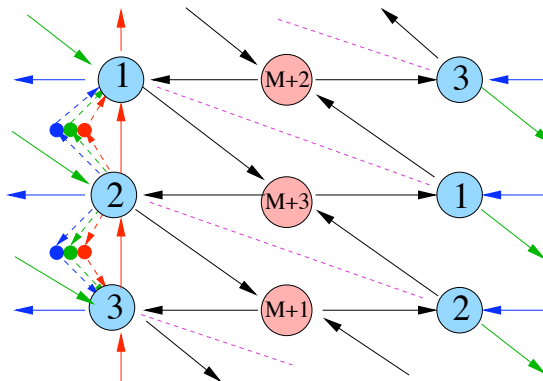
Figure 7. The quiver diagram of the SPP/$\mathbb{Z}_3$ orbifold gauge theory with flavor nodes. The mesons that acquire a non-zero vevs upon confinement of G are lightly indicated in magenta.

The result one finds is that, among the mesons formed out of the chiral matter charged under G (the black arrows in figure 7) only the mesons M_{yz}^{ii} and M_{zy}^{ii} that connect left and right SM nodes with the same rank acquire a vev. This leaves the diagonal SM gauge group unbroken. As in section 2 (and using the same notation) one finds that the mesons M_{yy}^{ij}, M_{yz}^{ij}, M_{zy}^{ij}, and the linear combination $\Phi^{ij} - M_{zz}^{ij}$ acquire a mass. The matter fields that remain light are X^{ij}, $\tilde{X}^{ij}$ and $\Phi^{ij} + M_{zz}^{ij}$, and all the flavored matter connected to the flavor nodes.

In other words, all colored lines in figure 7 directly reduce to light particles, except that the solid red arrows (the Φ fields) mix with the meson fields M_{zz}^{ij}, that are charged under the right copy of the SM gauge group. Physically, this means that all SM particles in this model are elementary (at least relative to this last confinement transition at the bottom of the RG cascade) except that one generation of the left-handed quark doublet and right-handed up quark is partially composite. From a phenomenological perspective, it is natural to identify the two elementary quark families with the first two generations, and the partially composite quarks with the third generation.[6]

4.2 The SPP/$\mathbb{Z}_3$ cascade

The duality cascade for SPP/$\mathbb{Z}_3$ was analyzed in detail in [14]. Starting from figure 5 at some UV point, it terminates in figure 6 as N is gradually reduced to zero. Similarly, M also decreases towards the IR. The cascade follows directly from the one in the parent SPP theory, which at each step dualizes the highest rank node out of the two without an adjoint [22]. After each dualization, the quiver becomes again the SPP one, with the adjoint field shifted to another node. Seiberg duality on a single SPP node translates to a dualization of all three nodes in a column of the SPP/$\mathbb{Z}_3$ quiver. The cascade then corresponds to sequentially dualizing the three columns.

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[6] Experimental bounds, from tests of $Z \to b\bar{b}$, only allow for a small amount of TeV-scale compositeness of b_L. As we will discuss later, we have other reasons to assume that the confinement scale Λ_c will be in the multi-TeV range.
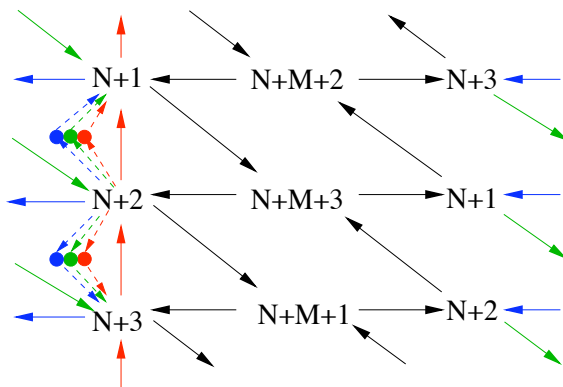
Figure 8. The quiver diagram of the SPP/$\mathbb{Z}_3$ orbifold gauge theory with flavor nodes, at some arbitrary point in the cascade.

After each dualization, the D7-branes 'move' over the quiver, i.e (anti)fundamental fields connect to new nodes [14]. When a node that contains (anti)fundamental fields is dualized, some of the resulting Seiberg mesons are singlets under all the gauge groups. These fields transform in bifundamental representations of the global symmetry nodes in the extended quiver and can be represented by arrows connecting them. These mesons, together with their evolution with scale, holographically encode degrees of freedom in D7-D7' sectors. They are generically necessary in flavored cascades in order to match anomalies in global symmetries. In this particular example, they become massive after a few steps and can be integrated out. In other cases, such as in the model studied in [47], their number can build up towards the UV.

5 RG running

Let us review the basic ideas necessary for computing duality cascades from a field theory perspective. Cascading theories are generically at strong coupling, with fields having $\mathcal{O}(1)$ anomalous dimensions. It is crucial to take into account these large anomalous dimensions (equivalently superconformal R-charges) to appropriately determine the running of couplings. The starting point is a superconformal quiver gauge theory with $\prod_{i=1}^{k} \mathrm{SU}(n_i)$, with $n^i = r^i N$.[7] To compute the anomalous dimensions in the superconformal theory, we first require the vanishing of the beta functions for gauge couplings

$$\beta_i = \frac{dx_i}{d\ln\mu} = 3\,n^i + \frac{1}{2}\sum_{j=1}^{k}(f_{ij}(\gamma_{ij}-1) + f_{ji}(\gamma_{ji}-1))n^j \tag{5.1}$$

with x_i given in (1.1). We present equations in a form that is most suitable for its application to quiver gauge theories. f_{ij} is the symmetric adjacency matrix of the quiver and gives the number of bifundamental multiplets between nodes i and j. The superpotential

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[7]Most of the examples considered in the literature correspond to $r_i = 1$ for all i.

is of the form $W = \sum_\mu h_\mu W_\mu$, with W_μ gauge invariant monomials. The couplings h_μ have scaling dimension

$$\beta(h_\mu) = -d(h_\mu) + \frac{1}{2} \sum_{X_{ij} \in W_\mu} \gamma_{ij} \tag{5.2}$$

where $d(h_\mu) = 3 - \deg(W_\mu)$ is the naive mass dimension of h_μ.

The discussion can equivalently be casted in terms of superconformal R-charges, which can be traded by anomalous dimensions by means of the usual relation $R_{ij} = 2/3 + \gamma_{ij}/3$. Typically, the equations above do not fully determine the anomalous dimensions. This is a manifestation of the fact that conformal invariance is not sufficient to rule out the possible mixing of the superconformal R-charge with additional U(1) global symmetries. The remaining freedom is fixed by a-maximization [35]. Within the space of solutions to (5.1) and (5.2), the R-charges must maximize the trial central charge

$$a = \frac{3}{32} \left[2 \sum_i n_i^2 + \sum_{i<j} f_{ij} n_i n_j [3(R_{ij} - 1)^3 - (R_{ij} - 1)] \right]. \tag{5.3}$$

Conformal invariance is then broken by introducing small numbers (compared to N) M and K of D5 and D7-branes.[8] From a field theory perspective, this corresponds to small modifications of the gauge group ranks and a small number of (anti)fundamental fields. In the presence of D5 and D7-branes, anomalous dimensions have an expansion

$$\gamma = \sum_{m,n=0}^{\infty} \gamma^{(m,n)} (M/N)^m (K/N)^n. \tag{5.4}$$

We will be later interested in computing the beta fuction of the total coupling (1.1), which we denote β_{tot}. In order to determine it to leading order, it is sufficient to consider the conformal values $\gamma_{i,j}^{(0,0)}$ (equivalently the associated R-charges). We justify this statement below.

When computing the beta functions for individual gauge couplings, we must add to (5.1) the contributions of the additional fundamental flavors. Combining all the beta functions, we obtain

$$\begin{aligned} \beta_{tot} &= \beta_{tot}^0 + \frac{3}{2} \sum_{D7_i} (R_{q_i} + R_{\tilde{q}_i} - 2) \\ &= \beta_{tot}^0 - \frac{3}{2} \sum_{D7_i} R_{X_i} \end{aligned} \tag{5.5}$$

β_{tot}^0 corresponds to the sum of contributions (5.1) for the original quiver, without extending it by global nodes. Notice that the ranks that enter in β_{tot}^0 are not only the result of the regular D3 and wrapped D5-branes but are also modified if flavor D7-branes are included. In the second line, X_i is a "mesonic" chiral operator (i.e. a product of chiral fields, corresponding to a path in the quiver, with internal color indices contracted) that transforms

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[8]Our discussion generalizes without changes to cases with more than one type of D5 and D7-branes.

in the conjugate representations of q_i and $\tilde{q}_i$ which has a superpotential coupling to flavors of the form

$$W \supset \tilde{q}_i X_i q_i. \tag{5.6}$$

In fact, every holomorphic D7-brane embedding is in one to one correspondence with a superpotential coupling of this type. In appendix A, we discuss this correspondence in more detail, with explicit examples for the case of $\mathbb{C}^3/\mathbb{Z}_3$. In (5.5), we have used that (5.6) has R-charge equal to 2.[9]

So far, the only thing we know is that β_{tot} is zero in the absence of flavor D7-branes, with the first non-trivial contribution potentially appearing at $\mathcal{O}(K)$. Both terms in (5.5) can a priori contribute. Since, as we have already mentioned, the ranks of gauge groups are modified in the presence of flavor D7-branes (in addition, this also changes the number of flavors each gauge group sees coming from bifundamentals connecting to other nodes) it is a in principle possible that β_{tot}^0 provides a non-vanishing contribution to the beta function.

Let us compute the linear $\mathcal{O}(K)$ term in β_{tot}. To calculate the contribution of β_{tot}^0 to this linear term, we can find the beta function for each gauge node and add them up.[10] We now introduce the additional input that the gauge theory we are considering is not an arbitrary quiver, but it is engineered with D-branes on a singularity. As explained, each node in the quiver corresponds to a bound state of D3, D5 and D7-branes (with the latter wrapped over compact 2 and 4-cycles, respectively). Since the D7-brane components do not extend radially, they do not contribute to a radial dependence of the dilaton. We conclude that $\beta_{\text{tot}}^0 = 0$. We are left with the simple expression

$$\beta_{\text{tot}} = -\frac{3}{2} \sum_{D7_i} R_{X_i}. \tag{5.7}$$

An interesting consequence of this result is that, even in the case $\gamma^{(1,0)}$ and/or $\gamma^{(0,1)}$ do not vanish, they do not contribute to β_{tot} to linear order. In section 6 we rederive this result from gravity.

5.1 Ignition scale

We now have all the tools necessary for estimating the decoupling scale Λ_c. As discussed in section 1, it is sufficient to follow x_{tot}, avoiding the intricate evolution of independent gauge couplings along the cascade.[11]

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[9]Once conformal invariance is broken, there is no longer a map between dimensions and R-charges of operators. To keep expressions short, we have chosen to express (5.5) in terms of R-charges rather than anomalous dimensions. The reader should be cautious and keep in mind that what we really mean is that we should write (5.5) in terms of anomalous dimensions, derive the result (5.7) concluding that corrections to anomalous dimensions do not enter β_{tot} to leading order and then switch to the more compact notation in terms of R.

[10]Notice that while we focus on the linear term of the beta function, we are *not* assuming that either $\gamma^{(1,0)}$ or $\gamma^{(0,1)}$ vanish.

[11]x_{tot} only involves the non-abelian part of the gauge group. It is possible to trace U(1) couplings along the entire RG-flow, but we do not consider them in our discussion.

Let us denote IR and UV the energy regimes below and above Λ_c, respectively. In the IR, the theory is weakly coupled so it is sufficient to compute $\beta_{\text{tot}}^{\text{IR}}$ at 1-loop. On the other hand, the theory is strongly coupled above Λ_c and $\beta_{\text{tot}}^{\text{UV}}$ is given by (5.7).

The gauge theory below Λ_c might contain additional charged fields beyond the matter content of the pure MSSM (e.g. four extra doublets in our example). These fields have to be heavier than roughly 1 TeV. Their effect on the RG-running is to reduce $\beta_{\text{tot}}^{\text{IR}}$ at energies above their masses, pushing the intersection scale Λ_c to higher energies. As a result, it is sufficient to take $\beta_{\text{tot}}^{\text{IR}}$ to be given by the MSSM result in order to get a lower bound on Λ_c. This corresponds to taking the unknown new masses to be $\mathcal{O}(\Lambda_x)$. In other words, computing the matching using the IR running of the pure MSSM provides a lower bound for new physics, which can be either the appearance of new strongly coupled gauge groups or the more standard possibility of exotic heavy matter.

Let us discuss in some more detail the behavior of x_{tot} around Λ_c. Following section 2, the G nodes confine and the ones in the two SM sets are pairwise higgsed to the diagonal SM. Since G is at infinite coupling, x_{tot} only receives contributions from the two SM sets. Furthermore, each of the two SM copies contains a U(1) node. We can think about them as containing trivial "SU(1)" factors with no associated gauge couplings. These nodes arise after dualizing $N_f = N_c + 1$ gauge groups. These Seiberg dualities take place when their inverse couplings vanish. We conclude that, just above Λ_c, x_{tot} only gets contributions from the two SU(2) and two SU(3) nodes in the SM copies. The inverse squared couplings of the diagonal groups are simply given by

$$x_i^{(SM_D)} = x_i^{(SM_L)} + x_i^{(SM_R)} \tag{5.8}$$

with $i = 2, 3$.

Let us proceed to the actual computation of Λ_c. First, we calculate the superconformal R-charges. Using the notation for fields in the parent SPP theory, we first see that $R_X = R_{\tilde{X}} = R_Y = R_{\tilde{Y}}$ and $R_Z = R_{\tilde{Z}}$ due to symmetries. Vanishing of the beta functions (5.1) and (5.2) for all gauge and superpotential couplings reduces to

$$\begin{aligned} R_\Phi + 2R_X &= 2 \\ R_X + R_Z &= 1 \end{aligned} \tag{5.9}$$

We have two independent equations in three variables. As we have explained, the remaining freedom is fixed by maximizing (5.3). The result is

$$R_\Phi = 2 - \frac{2}{\sqrt{3}} \qquad R_{Z,\tilde{Z}} = 1 - \frac{1}{\sqrt{3}} \qquad R_{X,\tilde{X},Y,\tilde{Y}} = \frac{1}{\sqrt{3}}. \tag{5.10}$$

Plugging these values into (5.7) we obtain $\beta_{\text{tot}}^{\text{UV}} = -9$. Below the ignition scale, we have $\beta_{\text{tot}}^{\text{IR}} = 2$. This value is obtained by computing the 1-loop beta functions for the SU(2) and SU(3) inverse squared gauge with the MSSM matter content and adding them up.

The ignition scale Λ_c is determined by matching the IR and UV running of x_{tot} as shown in figure 3. The boundary condition in the IR is given by the value of x_{tot} at some given scale. From [36], we obtain $x_2 \sim 185$ and $x_3 \sim 53$, i.e. $x_{\text{tot}} \sim 238$, at the Z-pole

$M_Z = 91.2 \, \text{GeV}$. The UV piece of the RG flow is parametrized by the position of the duality wall Λ_w, where $x_{\text{tot}} = 0$. Putting everything together, we get

$$x_{\text{tot}}^{\text{IR}} = 238 + 2\log\left(\Lambda/M_Z\right) \qquad x_{\text{tot}}^{\text{UV}} = -9\log\left(\Lambda/\Lambda_w\right) . \qquad (5.11)$$

Since the wall indicates the transition to the full 10-d string theory, it is natural for Λ_w to sit close to M_{Pl}. For $\Lambda_w = 10^{19} \, \text{GeV}$ we get $\Lambda_c \sim 3000 \, \text{TeV}$. We can contemplate lowering the wall to GUT scales. For $\Lambda_w = 10^{16} \, \text{GeV}$ we get $\Lambda_c \sim 10 \, \text{TeV}$. It is interesting to notice that the Λ_c/Λ_w hierarchy is mostly determined by IR parameters: namely the D7-branes that are necessary to obtain a sensible MSS-like model. In addition, even though the decoupling scale can get interesting low values, it naturally comes out sufficiently high to avoid any compositeness constraint.

6 Supergravity dual

We now turn to describe the properties of the supergravity gravity dual of a cascading gauge theory. The specific example of interest is the dual of the gauge theory associated with a collection of fractional and flavor branes at the $\mathbb{Z}_3$ orbifold of the suspended pinch point singularity. The SPP singularity is the real cone over the L^{121} space, one of the members of the L^{abc} series of 5-manifolds. The metric on this space is explicitly known [37, 38], and this in principle allows us to be rather explicit about the gravity dual of our specific example. We will try to keep our discussion somewhat general, however, using the SPP/$\mathbb{Z}_3$ example as our guide.

The 6-dimensional internal manifold, on which the various branes will be wrapped, is an orbifold of a cone over an Einstein-Sasaki 5-manifold Σ_5, which we assume is an L^{abc} space. The 5-manifold itself is a circle bundle over a 4-d base manifold Σ_4. The metric on the total 6-d cone takes the form

$$ds_6^2 = dr^2 + r^2(d\psi + A)^2 + r^2 ds_4^2 \,, \qquad (6.1)$$

where $ds_4^2 = h_{a\bar{b}} dz^a dz^{\bar{b}}$ is the metric of the 4-d Kähler-Einstein base manifold Σ_4, with $R_{a\bar{b}} = 6h_{a\bar{b}}$ and Kähler 2-form $J_{a\bar{b}} = 6iR_{a\bar{b}}$. The coordinate ψ is the angular coordinate on the circle bundle of the base Σ_4 and the one-form A satisfies

$$dA = 2J.$$

For the non-compact cone, the r-coordinate has infinite range. We assume that the orbifold is defined via isometric identification map on the base manifold Σ_5.

The base orbifold Σ_5/Γ contains various homology cycles, that may support wrapped D-branes and their associated RR-fluxes. Our brane configuration involves D-branes wrapped over both compact cycles within the base Σ_4 of the cone and non-compact cycles (flavor branes). Correspondingly, the world-volume gauge fields of the branes on compact cycles have normalizable zero modes and descend to local 4-d gauge fields, while for the flavor branes, the zero mode of the world-volume gauge fields is non-normalizable and the 7-d gauge group descends to a global flavor symmetry in 4-dimensions.

6.1 Gravity dual without D7-branes

To begin, let us summarize the asymptotic behavior of the solution dual to the cascading gauge theory associated with N D3-branes and M wrapped D5-branes, without any flavor D7-branes [12, 38–40]. As in the KS solution, it is characterized by the presence of the imaginary anti-self-dual 3-form flux.

The 4-d base manifold Σ_4 supports an anti-self-dual closed (1,1)-form ω_2, satisfying $\star_4\,\omega_2 = -\omega_2$. We normalize ω_2 to have unit period around the dual 2-cycle $\mathcal{C}_2$. It is then straightforward to show that the (2,1)-form

$$\Omega_{2,1} = \kappa\,\big(\frac{dr}{r} + i(d\psi + A)\big) \wedge \omega_2 \tag{6.2}$$

is imaginary self-dual and primitive. Here κ is a normalization constant, fixed such that $\Omega_{2,1}$ has unit period around its dual 3-cycle $\mathcal{C}_3$. With the help of this, we can now write the supersymmetric Ansatz for the asymptotic metric and RR-field strength [41]

$$ds^2 = \frac{1}{\sqrt{h}}\,dx_{1,3}^2 + \sqrt{h}\,ds_6^2\,,$$

$$g_s F_5 = (1 + \star_{10})dh^{-1}\wedge d^4x\,. \tag{6.3}$$

$$G_3 = iM\,\Omega_{2,1}\,, \qquad G_3 = dC_2 + \tau H_3$$

where $\tau = C_0 + ie^{-\phi}$ is the axio-dilaton, $F_p = dC_{p-1} - C_{p-3} \wedge H_3$ denote the RR p-form field strengths, and $H_3 = dB$ is the field strength of the NS 2-form B. The solution for the warp factor h follows from its relation with the self-dual 5-form field strength, which in turn is obtained by integrating its Bianchi identity

$$dF_5 = H_3 \wedge F_3\,. \tag{6.4}$$

The shape of the warp factor dictates the holographic identification between the radial coordinate r and the RG scale of the 4-dimensional gauge theory. The solution without D7-branes has a constant dilaton $e^\phi = g_s$ and axion C_0.

The above equations uniquely characterize the solution for large values of the radial coordinate r. The supergravity solution can be trusted in the regime where the 3-form flux M is large compared to $1/g_s$. Naive extrapolation to small r, however, yields a solution that develops a singularity at its tip. This singularity needs to be smoothed out via a suitable complex deformation, similar as for the KS solution. As explained in the introduction, however, for the specific application to the cascading theory of interest, we are interested in the large radius region only, corresponding to regime II far from its IR starting point. In this region, not only the supergravity approximation can be trusted (since curvature invariants are controlled by $g_s N$, which is large) but also the deformation is unimportant.

A D5-brane wrapped around the 2-cycle $\mathcal{C}_2$ creates one unit of F_3 flux around the dual 3-cycle $\mathcal{C}_3$. The solution without D7-branes has a fixed D5-brane charge M, read off by integrating the RR 3-form field strength over the 3-cycle $\mathcal{C}_3$ within Σ_5 dual to $\Omega_{2,1}$

$$M = \int_{\mathcal{C}_3} F_3\,. \tag{6.5}$$

The D3-brane charge, however, is not fixed: it evolves as a function of the holographic RG-coordinate r. We can introduce the effective D3-brane charge as the RR 5-form flux through the 5-d base manifold Σ_5

$$N^{\text{eff}} = \int_{\Sigma_5} F_5 \,. \tag{6.6}$$

F_5 is sourced by the D3-branes, and as seen from eq. (6.4), by the cross product of the 3-form fluxes. The r-dependence of N^{eff} can be expressed as

$$\frac{dN^{\text{eff}}}{dr} = M \, \frac{d\beta}{dr} \tag{6.7}$$

where

$$\beta = \int_{\mathcal{C}_2} B \tag{6.8}$$

denotes the period of the NS 2-form around the 2-cycle $\mathcal{C}_2$.

Using the expression (6.3) for the imaginary anti-selfdual 3-form, we read off

$$\beta(r) \simeq \kappa g_s M \log r \,. \tag{6.9}$$

As shown in the original work of Klebanov and Strassler, the r-dependence of the D3-brane charge precisely matches with the growth in the gauge group rank during the RG cascade of the dual gauge theory. The period β represents the difference between the inverse gauge couplings of gauge groups, and its r dependence reflects the running of these couplings. Evidently, the rate of growth in the total rank of the gauge groups during the RG cascade is proportional to the beta-functions that drives the gauge couplings. The beta function is constant and the effective rank thus growth logarithmically with scale.

6.2 Adding the flavor D7-branes

We now add the flavor D7-branes to this system. We will take into account their effect on the background to linearized order, in very much the same spirit of [42]. This is an allowed approximation in the supergravity regime II, where the D5-brane charge M is much larger than K, the number of D7-branes.

In Einstein frame, the D7-brane world volume action reads

$$\int_{D7} e^\phi \sqrt{- \det(\hat{g} + e^{-\phi/2} \, \mathcal{F})} \ + \int_{D7} \hat{C}_q \, e^{-\mathcal{F}} \tag{6.10}$$

It depends on the gauge field strength F via the combination

$$\mathcal{F} = F - B \,. \tag{6.11}$$

Flavor D7-branes are space-time filling. The supersymmetric branes wrap a holomorphic, non-compact 4-cycle within the 6-manifold, specified by some embedding equation

$$\alpha(z_i) = 0 \,, \tag{6.12}$$

with $\alpha(z_i)$ some holomorphic function defined on the internal 6-manifold. Although we will not attempt to do so here, it should in principle be possible to find the explicit form of the necessary embedding equations for the specific example of the SPP/$\mathbb{Z}_3$ quiver discussed in this paper. We will assume that $\alpha(z_i)$ is single-valued.

Since the D7-branes are a magnetic source for the axion field strength $F_1 = dC_0$, the axio-dilaton has non-trivial monodromy around the D7-brane locations. A holomorphic expression for the axio-dilaton with the proper monodromy is [43, 44]

$$\tau = \frac{i}{g_s} - \frac{i}{2\pi} \log \alpha(z_i). \tag{6.13}$$

Notice that when obtaining (6.13) we are just imposing that τ has the correct SL(2, Z) monodromy. We therefore expect this expression to be valid even beyond the linearized regime. Indeed, this is the case in known flavored examples where the full backreacted solution has been obtained [45–47]. If S^1 denotes a circle that surrounds K D7-branes, that is, if the function $\alpha(z_i)$ has K zeroes within the region surrounded by the S^1, then

$$\oint d\tau = \oint_{S^1} F_1 = K \tag{6.14}$$

The holomorphic function $\alpha(z_i)$ can be factorized into a product of K elementary functions $\alpha_k(z_i)$, each representing the embedding equation of a single elementary D7-brane. As previously explained, these embedding equations arise via the F-term equations $X_k = 0$ of the extra flavored matter associated with the intersection between the flavor and the fractional branes. This correspondence motivates the interpretation of the $\alpha_k(z_i)$ as the holographic wavefunctions of the operators X_k associated with the corresponding flavor node [48, 49]. We can thus relate the asymptotic dependence of the embedding function $\alpha(z_i)$, as a function of the radial coordinate r and angular coordinate ψ, to the sum of the scaling dimensions and R-charges of the operators X_k, via

$$\alpha(z_i) = z_1^\beta \, \tilde{\alpha}(x_i), \tag{6.15}$$

with

$$z_1 = re^{i\psi/3}, \tag{6.16}$$

x_i the four additional internal coordinates and

$$\beta = -\frac{3}{2} \sum_{D7_k} R_{X_k}. \tag{6.17}$$

Plugging this into the expression for the dilaton gives

$$e^{-\phi} = \frac{1}{g_s} - \frac{\beta}{2\pi} \log(r/r_0) \tag{6.18}$$

Here we recognize the RG dependence of the total gauge coupling $x_{\text{tot}} = \sum_k x_k$ of the quiver gauge theory found in the previous section. We will explain this correspondence in the next subsection.[12]

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[12]Note that the r.h.s. of (6.18) passes through 0 at some critical radius $r = r_{\text{wall}}$, indicating the possible presence of a duality wall. As explained before, we assume that the throat geometry is cut-off, and glued into a more complete 10-d string compactification, at a scale $r_{\text{decomp}} < r_{\text{wall}}$, thereby avoiding the presence of an actual duality wall.

The combined presence of fractional and D7-branes leads to some subtle modification in the equations for the higher RR-forms F_3 and F_5. Since the NS 3-form flux is turned on, its pull-back to the D7 worldvolume will be non-zero. This will in turn induce worldvolume flux $\mathcal{F}$, that sources the RR-forms. Taking this into account, one derives the following Bianchi identitites

$$
\begin{aligned}
dF_1 &= -\Omega_2 \\
dF_3 &= H_3 \wedge F_1 - \mathcal{F} \wedge \Omega_2 \\
dF_5 &= H_3 \wedge F_3 - \frac{1}{2}\mathcal{F} \wedge \mathcal{F} \wedge \Omega_2
\end{aligned}
\tag{6.19}
$$

Here Ω_2 denotes the delta-function 2-form localized on the D7-brane world-volume. The extra localized terms reflect the fact that the flux of $\mathcal{F}$ and $\frac{1}{2}\mathcal{F} \wedge \mathcal{F}$ give rise to extra induced D5-charge and D3-charge. The presence of the localized source terms can also be understood by requiring that the $SL(2,\mathbb{Z})$ covariant 3-form field $G_3 = F_3 + \tau H_3$ remains single-valued around the D7-branes.

More important for our purpose, however, is that the above equations show that, in the presence of D7-branes, the flux of the RR 3-form F_3 is no longer quantized or constant. We can introduce an effective D5-brane charge

$$
M^{\text{eff}} = \int_{\mathcal{C}_3} F_3
\tag{6.20}
$$

Here the integral is performed at a given radial location r. Let us suppose that the F_1 flux through the S^1 at this location r takes the quantized value K. It is then straightforward to derive that the effective brane charges can be expressed in terms of the period β of the NS B-field as

$$
\begin{aligned}
M^{\text{eff}} &= M + K\beta \\
N^{\text{eff}} &= N + M\beta + \tfrac{1}{2}K\beta^2
\end{aligned}
\tag{6.21}
$$

The radial dependence of the period β can be found by integrating the Bianchi equation for the $SL(2,\mathbb{Z})$ covariant 3-form

$$
dG_3 = d\tau \wedge H_3 \,.
\tag{6.22}
$$

In the linearized approximation, and using eqs. (6.2), (6.3) and (6.13), this gives

$$
dG_3 \simeq \frac{\kappa g_s M}{2\pi i} \frac{d\alpha}{\alpha} \wedge \frac{dr}{r} \wedge \omega_2
\tag{6.23}
$$

This equation may be integrated, with the result

$$
\beta(r) \simeq \kappa g_s M \left(\log r + \frac{\beta}{2\pi}(\log r)^2 + \dots \right)
\tag{6.24}
$$

where we used eq. (6.15).

The equations (6.24) and (6.21) summarize the holographic manifestation of the RG cascade of the dual quiver gauge theory, and show how the presence of the extra flavors accelerates the RG running and of the growth in the number of colors. We will now elaborate this correspondence, and make a more detailed comparison with the gauge theory description of the duality steps.

6.3 Geometric description of the duality cascade

We wish to recover the properties of the duality cascade of the quiver gauge theory from the string dual. To this end, we will make use of two different geometrical perspectives:

(i) **D-brane-probe perspective.** The cascading gauge theory has not yet reached the large N regime. It can be engineered from fractional and flavor D7-branes on a singular cone geometry. Coupling constants are parametrized by the geometric data defined on the cone. RG-running is represented by adjusting these geometric data.

(ii) **Supergravity perspective.** The cascading gauge theory is assumed to be in the large N and large 't Hooft coupling regime. All degrees of freedom and couplings of the gauge theory can be captured by a dual geometry, with a number of flavor D7-branes. The fractional brane charge is carried by RR-flux, as summarized in section 6.2.

The supergravity dual correctly encodes global properties of the cascading quiver gauge theory, such as the RG running of couplings and overall growth in the number of colors, but does not appear to have sufficient structure to fully reconstruct the sequence of Seiberg duality maps. The D-brane probe picture, on the other hand, provides an attractive dual perspective on Seiberg duality, and thereby gives a useful interpolation between the gauge theory and supergravity description of the cascade.

How does one recover the duality cascade from the D-brane probe perspective? In what follows, we generalize the concept of *fractional branes* from the common use in the literature, and apply it to *any supersymmetric bound state of D3, D5 and D7-branes* wrapped over appropriate (not necessarily compact) cycles. This general approach allows us to study the 'color' D-branes (wrapping compact cycles within the 4-d base Σ_4 of the complex cone) and the flavor branes (whose D7-brane components wrap non-compact 4-cycles) on an equal footing.[13] The presentation below is complementary to closely related discussions in [46, 47].

The fluxes of F along the non-trivial 2-cycles in the base represent the D5-brane wrapping charges of the fractional brane; the D3-brane charge is the instanton number. The D7, D5 and D3 wrapping numbers are combined in a charge vector, which we denote by

$$\mathcal{Q} = (q_3, q_5, q_7) \,. \tag{6.25}$$

The previous expression, as the ones that follow, is schematic. $\mathcal{Q}$ is in general a multi-dimensional vector whose dimension is equal to the number of independent fractional

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[13]Although the flavor nodes themselves do not undergo any Seiberg dualities, the flavor D7-branes do influence and actively participate in the duality cascade, as we will see below.

branes. Following our generalized definition of fractional branes, this number is the sum of the total dimension of the cohomology groups of the base Σ_4 and the number of non-compact 4-cycles in the full cone. The quiver gauge theory is engineered from a collection of several intersecting fractional and flavor branes, with total charge

$$\sum_i N_i \, \mathcal{Q}_i \, = \, (N, M, K) \tag{6.26}$$

Here N_i denotes the multiplicity of the $\mathcal{Q}_i$ fractional brane, and equals the number of colors (or flavors) of the corresponding gauge (or global symmetry) group factor.

The gauge coupling of a fractional brane with charge vector $\mathcal{Q}_i$ is given by [50]

$$x_i = e^{-\phi} |Z(\mathcal{Q}_i)| \tag{6.27}$$

where $Z(\mathcal{Q}_i)$ is the central charge of the brane. It depends linearly on the charge vector

$$Z(\mathcal{Q}) = q_3 + q_5 \Pi + q_7 \mathcal{K}. \tag{6.28}$$

The formula (6.27) can be derived from the boundary CFT description of fractional branes, and does not rely on the supergravity approximation. In the geometric regime, the explicit form of the central charge can be found by expanding the Born-Infeld world-brane action (6.10). One finds that Π and $\mathcal{K}$ are given by the period integrals

$$\Pi = \int_{\mathcal{C}_2} (B + iJ) \,, \qquad \mathcal{K} = \frac{1}{2} \int_{\Sigma_4} (B + iJ)^2 \,, \tag{6.29}$$

where J denotes the Kähler 2-form of the 4-d base manifold Σ_4. The holographic RG running of the gauge couplings is driven by the evolution of these integrals and the dilaton along the radial direction. Note that the gauge coupling vanishes for the flavor D7-branes that wrap non-compact 4-cycles.

The central charge $Z(\mathcal{Q})$ specifies which $\mathcal{N} = 1$ sub-algebra of the $\mathcal{N} = 2$ supersymmetry of the IIA supergravity background is preserved by the fractional brane. A collection of branes is supersymmetric if all their central charges are aligned, and have the same sign.[14] This condition naturally dictates the appearance of a duality cascade. As the periods Π and $\mathcal{K}$ evolve under the holographic RG, the central charge vector of one or more branes may become zero and change sign at some value of the radial coordinate. Beyond this point, the original configuration of branes becomes unstable, and has to rearrange itself to realign all central charges. This rearrangement is the geometric manifestation of the Seiberg duality map.

Consider the location where the central charge of the $\mathcal{Q}_j$ brane changes sign. Roughly speaking, it then turns into its own anti-brane. To restore supersymmetry, and stability, the fractional branes must form a new set of bound states. After this rearrangement, the charge vector of the j-th fractional brane has changed sign

$$\mathcal{Q}_j \to -\mathcal{Q}_j \tag{6.30}$$

___

[14]A relative phase between central charges of two fractional branes translates into an FI-parameter of the gauge theory. Misalignment of the charges reflects supersymmetry breaking via D-terms.

and, as suggested by the Seiberg duality map, has a new positive multiplicity equal to

$$N_j \; \rightarrow \; F_j - N_j \tag{6.31}$$

where F_j denotes total number of flavors of the $\mathcal{Q}_j$ node. We can express F_j as

$$F_j = \sum_{i>j} N_i\, f_{ij} \tag{6.32}$$

where the sum runs over all other fractional branes that connect to the $\mathcal{Q}_j$ fractional brane via a positive number f_{ij} of incoming lines. The number f_{ij} equals the intersection number between the corresponding pair of fractional branes.

To absorb the overall change in the charge vector, these other branes form an appropriate bound state with $\mathcal{Q}_j$ branes, adjusting their charge vector according to[15]

$$\mathcal{Q}_i \rightarrow \mathcal{Q}_i - f_{ij}\, \mathcal{Q}_j\,. \tag{6.33}$$

From the geometric rules that give the chiral mater content, one easily verifies that the above re-combination of bound states reproduces the Seiberg duality map. More detailed discussions of this correspondence can be found in [51, 52]. Here we just make a few comments, that are relevant for the present discussion:

(i) From this adiabatic probe brane perspective, the Seiberg duality is not some similarity transformation, but a true dynamical readjustment of D-brane bound states.[16]

(ii) The full set of charge vectors and multiplicities satisfy the constraint (6.26): the total brane charge is preserved throughout the duality cascade.

(iii) The global flavor nodes have vanishing gauge coupling, and do not undergo any Seiberg duality. However, they do actively participate in the cascade: their presence modifies the duality rules for the other nodes, and their charge vectors transform according to (6.33). The flavor branes thus necessarily carry a composite D-brane charge.

At first sight, the second fact (ii) sounds somewhat surprising, and seemingly at odds with the usual argumentation for how the supergravity background mirrors the duality cascade. After all, after many duality steps towards the UV, all gauge group factors have acquired some large rank, and this should translate into a large total D-brane charge. Moreover, in this regime, the gauge theory is known to admit a dual supergravity description, supported by a correspondingly large RR-flux.

The apparent contradiction is resolved by taking into account the third remark (iii): the flavor nodes, while seemingly passive observers to the RG cascade, do accumulate D-brane charge along the way. In the probe brane perspective, this follows from (6.33). In

---

[15]The index j is not summed over. Note that the bound state formation changes the expression (6.27) for the gauge couplings of the nodes for $i > j$. The *value* of the coupling is continuous, since the transition (6.33) occurs at the specific radial location where $Z(\mathcal{Q}_j) = 0$.

[16]In the mathematical literature, the stable basis of fractional branes is called an 'exceptional collection', and the Seiberg duality map is a special map known as a 'mutation' [53, 54].

the supergravity, this effect is manifest from the D7-brane worldvolume action (6.10): its Chern-Simons term carries an *effective* D5 and D3-brane charge given by the flux of $\mathcal{F}$ and of $\frac{1}{2}\mathcal{F} \wedge \mathcal{F}$ through the corresponding cycles. This effective charge grows with the period β of the NS 2-form, and thus runs over a continuous range of values. This continuous effective charge is related to the 'true' quantized D-brane charge, given by the integral flux of F and $\frac{1}{2}F \wedge F$, as follows.

The D7 world-volume action is invariant under the redefinition $B \to B + d\Lambda$, $A \to A + \Lambda$, with A the world-volme gauge field and Λ any one-form. Λ is allowed to multi-valued, provided that $d\Lambda$ belongs to an integral cohomology class. Large gauge transformations of this type shift the period of B and the fluxes of F by an integer. Assuming that they simultaneously act on all fractional brane world-volumes, these large gauge transformation act on the individual and total D-brane charges via

$$
\begin{aligned}
\beta &\to \beta - n & q_7 &\to q_7 \\
M &\to M + nK & q_5 &\to q_5 + n\,q_7 \\
N &\to N + nM + \tfrac{1}{2}n^2 K & q_3 &\to q_3 + nq_5 + \frac{1}{2}n^2\,q_7
\end{aligned}
\tag{6.34}
$$

We can use this invariance to turn β into an angular variable, restricted to the interval between 0 and 1. Note, however, that the discrete gauge transformation (6.34) leaves the individual multiplicities N_i and intersection numbers f_{ij} between the fractional branes invariant. Hence (6.34) does not act on the gauge theory data.

We can thus summarize the geometric manifestation of the duality cascade as follows. The presence of H_3 flux implies that the period β steadily grows along the holographic RG-direction. Via eq. (6.27), this evolution matches with the RG running of the difference between the inverse gauge couplings x_i of the gauge group factors. In the process, *two distinct types of discrete transitions take place.* First, whenever one of the central charges $Z(\mathcal{Q}_i)$ changes sign, the fractional branes rearrange according to the Seiberg duality map (6.30)-(6.33). Secondly, whenever β becomes equal to 1, we may reset it to 0 by performing the integral shift (6.34). The former transformation leaves the total D-brane charge invariant, but changes the D-brane charge vectors, multiplicities (number of colors) and intersection numbers (number of flavors), according to the Seiberg duality map. The latter transformation does not act on the gauge theory data, but changes the total D3 and D5-brane charge in a way that reflects the accumulative growth of the gauge group ranks of the quiver gauge theory. At a phenomenological level, this growth is accurately captured by the continuous effective charge (6.21).

6.4 Total gauge coupling and the dilaton

Finally, let us discuss in some more detail why it is natural in this context to consider the total gauge coupling and explain its relationship with the dilaton. The argument is as follows. In region II, after the cascade has proceeded through several cycles upwards, the multiplicities N_i of the fractional branes all become large, and of the following form

$$
N_i = N + r_i\,M + s_i\,K \,,
\tag{6.35}
$$

where r_i and s_i are constants that encode the modification of the ranks in the presence D5 and D7-branes. From (6.34), we see that the number D7-branes remains constant while the D5-branes grow linearly with the step of the cascade n towards the UV. At the same time, the number of D3-branes grows quadratically with n. Then, we soon reach a regime in which $N \gg M, K$, and the effect of the fractional brane charge M and D7-brane charge K can be treated as small perturbations away from the conformal large N theory of N D3-branes. As a result, the central charge (6.27) is dominated by the D3-branes. From the linear relation (6.27)–(6.28) between the gauge coupling and the charge vector of the associated brane, we see that the total gauge coupling

$$x_{\text{tot}} = \sum_i x_i \tag{6.36}$$

in this quasi-conformal regime reduces with the help of eq. (6.26) to the anticipated result

$$x_{\text{tot}} = e^{-\phi}. \tag{6.37}$$

7 Conclusions

We have outlined the general principles of N-ification, a new class of string (motivated) UV extension of the SM. N-ification models provide a precise identification of the new degrees of freedom corresponding to embedding the SM in a warped higher dimensional space and their interaction with the SM: they are holographically mapped to a growing tower of field theory degrees of freedom in a duality cascade.

The N-ification program can also be viewed as the generation of a UV string theory starting from an IR gauge theory. As such, it can be applied to more general IR starting points, such as the MSSM augmented by some hidden sector. The complex deformation that terminates the cascade separates them in the extra dimensions. The two sectors are then coupled by higher dimension operators suppressed by the mass scale of string states stretching between them. Interesting hidden sectors include SUSY breaking ones as well as conformal field theories. The latter provide realizations of the unparticle scenario in N-unified models [55]. In fact, some of the models classified in [14] are explicit examples of this idea.

Flavor D7-branes are a usual ingredient in type IIB local constructions of MSSM-like theories. As we have explained, they lead to an acceleration of duality cascades. This is a welcome effect in a holographic realization of the SM, since it rapidly puts us in a large-N regime. The acceleration can be understood from both a field theory and a gravity dual perspective by simply following the evolution of the total gauge coupling and the dilaton, respectively. It leads to a large hierarchy between Λ_c (the scale at which the cascade is ignited) and a UV scale at which the theory transitions to the full 10-d string theory (which is naturally located close to the Planck scale). Interestingly, this hierarchy is determined by IR data — the number and type of D7-branes — rather than allowing arbitrary values, associated with some choice of fluxes, as in other constructions.[17]

---

[17]There also is some dependence on the extra gauge group G, which controls the extension of region I in figure 3. This dependence vanishes in the limit of large rank for G.

It would be interesting to investigate alternatives for SUSY breaking besides the possibility of a hidden sector. For example, whether the strong dynamics of the gauge group G might be exploited for this purpose.

A natural question is whether an N-ification extension of the SM can not only accommodate but also constraint the SM gauge couplings to have the observed values that are alternatively interpreted as arising from a GUT at high energies. In this work we have only used very general properties of the duality cascade, such as the running of the total gauge coupling. It is possible that the requirement of a periodic cascade imposes strict constraints on the allowed couplings at Λ_c . Not surprisingly, the simple analysis of [11] shows that in some cases periodic cascades rule out entire regions in the space of couplings.

Acknowledgments

A   D-branes on $\mathbb{C}^3/\mathbb{Z}_3$

In this section we discuss massless SUSY D7-brane embeddings in $\mathbb{C}^3/\mathbb{Z}_3$. A similar discussion applies to flavor branes in other singularities.

D3-branes on $\mathbb{C}^3/\mathbb{Z}_3$.   The simplest example of a three node quiver theory is that of single D3-brane probe near a $\mathbb{C}^3/\mathbb{Z}_3$ singularity. The orbifold identification gives rise to three 'image branes', that are interchanged under the $\mathbb{Z}_3$-action

$$(x_1, x_2, x_3) \rightarrow (\omega x_1, \omega x_2, \omega x_3), \qquad \omega^3 = 1. \qquad (A.1)$$

Open strings in the untwisted sector reconnect with the same brane, while the twisted sector open strings connect two different image branes. In the limit where the D3-brane approaches the orbifold fixed point, the three image branes recombine into three so-called fractional branes, corresponding to the three irreducible representations of the abelian orbifold group $\mathbb{Z}_3$.

For N D3-branes at the orbifold, each fractional brane will occur with multiplicity N. The world-volume quiver gauge theory has gauge group $U(N)^3$ and three 'generations' of bifundamental chiral matter, associated with the intersections between the fractional branes, as indicated in figure 9. The superpotential of the orbifold theory is the one inherited from the $\mathcal{N} = 4$ SYM covering theory, $W = \epsilon_{ijk}\text{Tr}(X^i X^j X^k)$.

In D-brane engineering of quiver gauge theories, the individual fractional branes can be used as independent building blocks. By assigning each fractional brane its own multiplicity
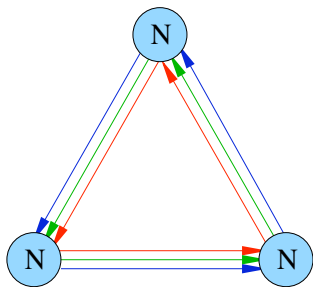
Figure 9. The quiver diagram of N D3-branes on a $\mathbb{C}^3/\mathbb{Z}_3$ orbifold.
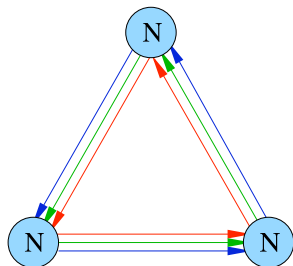
Figure 10. The three classes of flavor D7-branes on $\mathbb{C}^3/\mathbb{Z}_3$. Notice the opposite orientation between (a) and (b). While class (a) couple to single bifundamentals, class (b) couple to linear combinations of products of two of them. Class (c) are completly non-chiral.

n_i, one obtains a quiver theory with gauge group $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3)$. To get the Standard Model gauge group, we introduce six fractional branes, and let the $\mathbb{Z}_3$ orbifold symmetry act via the Chan-Paton matrices

$$\gamma_{D3} = \mathrm{diag}(\mathbb{1}_1, \omega\mathbb{1}_2, \omega^2\mathbb{1}_3). \tag{A.2}$$

In other words: one fractional brane is in the identity representation $\mathbb{1}$ of $\mathbb{Z}_3$, two branes are in the $\omega\mathbb{1}$ representation, and three branes are in the $\omega^2\mathbb{1}$ representation. This choice is invariant under $\mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{U}(3)$ gauge symmetry. When done in isolation, the particular choice (A.2) of Chan-Paton matrices would be inconsistent: the multiplicities N_i must satisfy the constraint that the resulting gauge theory is free on non-abelian anomalies. To arrive at a self-consistent construction, it is necessary introduce additional branes that, via their intersections with the fractional D3-branes, give rise to the other chiral matter fields that contribute to anomaly cancelation in the Standard Model. These extra branes are the flavor D7-branes. From a string theory point of view, the flavor D7-branes are required in order to cancel twisted RR tadpoles.

D7-branes in $\mathbb{C}^3/\mathbb{Z}_3$. D7-branes span the 3+1-dimensional space-time and a 4-dimensional subspace of the internal target space. Supersymmetry requires that this internal subspace is some holomorphic 4-cycle, specified by an embedding equation of the

form

$$f(z_i) = 0 \tag{A.3}$$

with $f(z_i)$ some holomorphic function defined on $\mathbb{C}^3/\mathbb{Z}_3$. The simplest class of elementary D7-branes are given by the embedding equations

$$z_r = X_{12}^r X_{23}^r X_{31}^r = 0 \,. \tag{A.4}$$

This equation evidently factorizes into three separate equations

$$X_{12}^r = 0 \quad \cup \quad X_{23}^r = 0 \quad \cup \quad X_{31}^r = 0 \,. \tag{A.5}$$

That is, eq. (A.4) is implied, if one of the three eqs. (A.5) is satisfied.

The flavor D7-branes intersect with the fractional branes and this produces extra chiral matter fields Q, which arise as the ground states of the D3/D7 sector of open strings. The embedding equation (A.4) arises as a solution to the F-term equation obtained by varying the superpotential for the chiral fields Q. Each separate factor in (A.5) is associated to a superpotential term

$$W = \tilde{Q}_i X_{i,i+1}^r Q_{i+1} \,. \tag{A.6}$$

The factorized solution of the embedding equation indicates that the D7-brane in fact divides into three components. Geometrically, eqs. (A.5) are the condition that the D3-branes are located within one of these three components[18], so that the D3/D7 open strings have massless ground states. The three components of the D7-brane are distinguished via the action of the $\mathbb{Z}_3$-transformations on the D7 Chan-Paton index. (the extra chiral matter fields transform under $\mathbb{Z}_3$ as $Q \to \omega Q, \tilde{Q} \to \omega \tilde{Q}$). This class of factorizable embedding is equivalent to the Ouyang embedding of D7-branes in the conifold [42].

Compound D7-branes are specified via an embedding equation of the form

$$X_{12}^r (X_{23}^{s_1} X_{31}^{t_1} + X_{23}^{s_2} X_{31}^{t_2}) = 0 \,, \tag{A.7}$$

with $s_1 \neq s_2$ and $t_1 \neq t_2$. Eq. (A.7) factorizes as

$$X_{12}^r = 0 \qquad \cup \qquad X_{23}^{s_1} X_{31}^{t_1} + X_{23}^{s_2} X_{31}^{t_2} = 0 \,. \tag{A.8}$$

The second equation can not be factorized further, and arises as the F-term equation of a quartic superpotential term

$$W = \tilde{Q}_2 (X_{23}^{s_1} X_{31}^{t_1} + X_{23}^{s_2} X_{31}^{t_2}) Q_1 \tag{A.9}$$

We have set the coefficients of both terms in (A.7) to 1. By tuning their ratio, we can make the embedding (and consequently the gauge theory) arbitrarily close to one with a single term in the superpotential. In that case, the second term becomes important only for very large vevs of the corresponding fields. The non-factorizable embeddings are analogous to Kuperstein embedding in the conifold [56].

---

[18]A D3-brane and D7-brane can form a supersymmetric bound state, when the D3-brane is embedded inside the D7 worldvolume.

There are 27 gauge invariant mesonic operators $a^{\alpha\beta\gamma} = X_{12}^{(\alpha)} X_{23}^{(\beta)} X_{31}^{(\gamma)}$ in this theory.[19] Following the previous discussion, they can be used to define D7-branes of the three general classes shown in figure 10. The D7-branes in figure 10.c can be constructed with a straightforward extension of the above reasoning. They correspond to completely non-factorizable cubic embeddings.

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[19]It is very useful to use the SU(3) global symmetry of the theory to organize calculations [57]. The $X_{ij}^{(\alpha)}$ transform in the fundamental representation. The $a^{\alpha\beta\gamma}$ transform in $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$. Using the F-term equations, we conclude that any antisymmetric combination of the indices vanishes and only the 10 survive. We are not going to exploit this clasification here.

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